JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **88**, 257–264 (1982)

A Note on Ladas' Paper: Oscillatory Effect of Retarded Actions

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Submitted by C. L. Dolph

1. INTRODUCTION

Recently Ladas [2] considered the equation

$$x^{(2n)} - q(t)\,x(t-\tau) = 0, \qquad n \geqslant 1, \tag{1}$$

where $q \in C[[0, \infty), [0, \infty)]$, $q(t) \neq 0$, and τ is a positive constant and proved the following theorem:

THEOREM (*). *Assume that for every $t_0 > 0$ there exists $t > t_0$ such that*

$$\int_{t-\tau}^{t} [(t-s)^{2n-1} q(s+\tau) - (t-s-\tau)^{2n-1} q(s)]\, ds \geqslant (2n-1)! \tag{H_1}$$

and

$$\int_{t-\tau}^{t} [(t-s)^k q(s+\tau) - (t-s-\tau)^k q(s)]\, ds \geqslant 0 \tag{H_2}$$

$$\text{for} \quad k = 0, 1, \ldots, 2n-2.$$

Then every bounded solution of (1) *is oscillatory.*

The purpose of this note is to establish oscillatory character for a larger class of solutions of the equation, namely

$$L_n x(t) + (-1)^{n+1} q(t)\, x(t-\tau) = 0, \qquad n \geqslant 1, \tag{2}$$

where

$$L_0 x(t) = x(t), \qquad L_k x(t) = a_k(t)(L_{k-1} x(t))^{\cdot} \qquad \left(\cdot = \frac{d}{dt}\right),$$

0022-247X/82/070257-08$02.00/0

for $k = 1, 2, \ldots, n$. We assume that $a_0 = a_n = 1$,

$$q, a_i \in C[[0, \infty), (0, \infty)], \quad \int^{\infty} \frac{1}{a_i(s)}\, ds = \infty, \quad \text{for } i = 1, 2, \ldots, n-1$$

$$\lim_{t \to \infty} \frac{1}{\alpha_1(t)} \sum_{i=0}^{k} c_i \alpha_i(t) > 0, \qquad \alpha_0(t) = 1,$$

for every choice of the constants c_i with $c_k > 0$, $k = 1, 2, \ldots, n-1$, where

$$\alpha_1(t) = \int_c^t \frac{1}{a_1(s)}\, ds, \quad \alpha_2(t) = \int_c^t \frac{1}{a_1(s)} \int_c^s \frac{1}{a_2(u)}\, du\, ds,$$

and

$$\alpha_k(t) = \int_c^t \frac{1}{a_1(s_1)} \int_c^{s_1} \cdots \int_c^{s_{k-1}} \frac{1}{a_k(s_k)}\, ds_k \cdots ds_1,$$
$$k = 3, \ldots, n-1, \quad t \geqslant c \geqslant 0.$$

A nontrivial solution of (2) is said to be oscillatory on $[0, \infty)$, if it has an infinity of zeros, otherwise it is said to be nonoscillatory on $[0, \infty)$.

Our main results are given in Section 2 (Theorems 1 and 2), which generalize and improve the results of [2] and some of the results in [3, 4]. Our results when specialized to (1) turn out to be a substantial improvement of Theorem 1 in [2] and Theorem 2.1 in [3].

2. Main Results

Let $1 \leqslant k \leqslant n-1$ and $t, s \in [0, \infty)$. We define

$$w_1(t, s) = \int_s^t \frac{1}{a_1(u)}\, du \quad \text{and} \quad w_k(t, s) = \int_s^t \frac{1}{a_k(u)} w_{k-1}(u, s)\, du.$$

Lemma 1. *If $x(t)$ is a solution of* (2), *then for all $t, s \in [0, \infty)$*

$$x(s) = \sum_{j=0}^{n-1} (-1)^j w_j(t, s) L_j x(t) + \int_s^t w_{n-1}(u, s)\, q(u)\, x(u - \tau)\, du,$$
$$w_0(t, s) = 1. \tag{3}$$

Proof. The proof of Lemma 1 is by induction and will be omitted.

Lemma 2. *If $x(t)$ is a nonnegative and nonincreasing solution of* (2), *then for $t - \tau \leqslant s \leqslant t$, the following inequality holds*:

$$x(s)\left[1 - \int_s^t w_{n-1}(u, s) q(u)\, du\right] \geqslant \sum_{j=0}^{n-1} (-1)^j w_j(t, s) L_j x(t). \tag{4}$$

Proof. If $s \leqslant u \leqslant t$, then $s-\tau \leqslant u-\tau \leqslant t-\tau \leqslant s$ and therefore $x(u-\tau) \geqslant x(s)$. Using this inequality in (3), inequality (4) follows immediately.

The following Lemma is an analog of a result due to Kim [1], so we omit the details.

LEMMA 3. *If x is a nontrivial solution of (2) such that $x(t-\tau) \geqslant 0$, $x(t) \geqslant 0$, and $x(t)/a_1(t) \to 0$ as $t \to \infty$, then $x(t) > 0$, $\dot{x}(t) < 0$, $(-1)^k L_k x(t) > 0$ for $k = 2, 3, \ldots, n-1$, $t \in [0, \infty)$, and $L_k x(t) \to 0$ monotonically as $t \to \infty$, and $k = 1, 2, \ldots, n-1$.*

LEMMA 4. *If $x(t)$ is a nonnegative and nonincreasing solution of (2) for $t-\tau \leqslant s \leqslant t$, the following inequality holds:*

$$\begin{aligned}
&(-1)^n L_{n-1} x(t+\tau) \\
&\quad \geqslant (-1)^{n-1} L_{n-1} x(t) \left[\int_{t-\tau}^{t} w_{n-1}(t,s) q(s+\tau)\, ds - A_0(t-\tau) \right] \\
&\qquad + \sum_{j=0}^{n-2} (-1)^j L_j x(t) \left[\int_{t-\tau}^{t} w_j(t,s) q(s+\tau)\, ds + (-1)^{j+1} A_{n-j-1}(t-\tau) \right] \\
&\qquad - \int_{t-\tau}^{t} x(s+\tau) A_n(s)\, ds, \qquad (5)
\end{aligned}$$

where

$$A_0(s) = 1 - \int_s^t w_{n-1}(u,s)\, q(u)\, du$$

and

$$A_k(s) = a_{n-k}(s+\tau)(A_{k-1}(s))^{\cdot}, \qquad k = 1, 2, \ldots, n.$$

Proof. Multiplying both sides of (4) by $q(s+\tau)$ and using (2), we obtain

$$\begin{aligned}
&(-1)^n L_n x(s+\tau) \left[1 - \int_s^1 q(u) w_{n-1}(u,s)\, du \right] \\
&\qquad \geqslant \sum_{j=0}^{n-1} (-1)^j q(s+\tau) w_j(t,s)\, L_j x(t). \qquad (6)
\end{aligned}$$

Set

$$A_0(s) = F(s) = 1 - \int_s^t q(u) w_{n-1}(u,s)\, du.$$

Integration by parts yields

$$\int_{t-\tau}^{t} L_n x(s+\tau) F(s)\,ds = \sum_{j=0}^{n} (-1)^{j+1} L_{n-j} x(s+\tau) A_{j-1}(s) \Big|_{t-\tau}^{t} + (-1)^n \int_{t-\tau}^{t} A_n(s) x(s+\tau)\,ds.$$

Now inequality (5) follows by integrating both sides of (6) with respect to s from $t-\tau$ to t.

In the following Theorem we consider (2) with n replaced by $2n$; namely,

$$L_{2n} x(t) - q(t) x(t-\tau) = 0, \qquad n \geqslant 1. \tag{7}$$

THEOREM 1. *Assume that for every $t_0 > 0$, there exists $t > 0$ such that the following hypotheses are satisfied*:

$$\int_{t-\tau}^{t} q(s+\tau) w_{2n-1}(t,s)\,ds - A_0(t-\tau) \geqslant 0, \tag{8}$$

$$\int_{t-\tau}^{t} q(s+\tau) w_j(t,s)\,ds + (-1)^{j+1} A_{2n-j-1}(t-\tau) \geqslant 0, \tag{9}$$

$$j = 0, 1, \ldots, 2n-2,$$

and

$$A_{2n}(t) \leqslant 0 \qquad \text{for} \quad t \geqslant t_0. \tag{10}$$

Then every solution $x(t)$ of (7) *with the property that $x(t)/\alpha_1(t) \to 0$ as $t \to \infty$ is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (7) such that $\lim_{t\to\infty}(x(t)/\alpha_1(t)) = 0$. Without any loss of generality we assume that $x(t) > 0$ for $t \geqslant t_0$. By Lemma 3 there is a $t_1 \geqslant t_0$ such that $\dot{x}(t) < 0$ and $(-1)^k L_k x(t) > 0$ for $t \geqslant t_1$ and $k = 2, 3, \ldots, 2n-1$.

By replacing n by $2n$ in Lemma 4, we obtain

$$\begin{aligned} L_{2n} x(t+\tau) \geqslant & - L_{2n-1} x(t) \left[\int_{t-\tau}^{t} w_{2n-1}(t,s) q(s+\tau)\,ds - A_0(t-\tau) \right] \\ & + \sum_{j=0}^{2n-2} (-1)^j L_j x(t) \\ & \times \left[\int_{t-\tau}^{t} w_j(t,s) q(s+\tau)\,ds + (-1)^{j+1} A_{2n-j-1}(t-\tau) \right] \\ & - \int_{t-\tau}^{t} (x(s+\tau)) A_{2n}(s)\,ds. \end{aligned}$$

In view of hypotheses (8)–(10) there exists a $t > t_1$ such that the right-hand side (11) is nonnegative while its left-hand side is negative. This contradiction completes the proof.

Remark. If $a_i(t) = 1$ for $i = 1, \ldots, n$, then Eq. (7) reduces to (1), and if (1) possesses a bounded solution then it does satisfy the property $x(t)/t \to 0$ as $t \to \infty$. (Here

$$\alpha_1(t) = \int_0^t \frac{1}{a_1(s)}\, ds = \int_0^t ds = t).$$

Thus Theorem 1 is a considerable improvement of Theorem (*).

EXAMPLE 1. Consider the equation

$$\text{(a)} \qquad (t^{-1}\dot{x})^{\cdot} - x(t-\tau) = 0, \qquad t \geqslant \tau^2 + \tfrac{1}{6}\tau.$$

$$\begin{aligned} A_0(s) &= 1 - \int_s^t w_1(u, s)\, q(u)\, du \\ &= 1 - \tfrac{1}{2}(\tfrac{1}{3}t^3 - s^2 t + \tfrac{2}{3}s^3), \\ A_1(s) &= q_1(s+\tau) A_0^{\cdot}(s) = \frac{st - s^2}{s+\tau}, \end{aligned}$$

$$A_2(s) = A_1^{\cdot}(s) = -1 + \frac{\tau(t+\tau)}{(s+\tau)^2} \qquad \text{so} \quad A_2(t) \leqslant 0$$

$$\text{for} \quad t > 0.$$

Thus

$$A_0(t-\tau) = 1 - \tfrac{1}{2}t\tau^2 + \tfrac{1}{3}\tau^3 \quad \text{and} \quad A_1(t-\tau) = \tau - (\tau^2/t).$$

The hypotheses (8)–(10) are verified, since

$$\int_{t-\tau}^t \tfrac{1}{2}(t^2 - s^2)\, ds - (1 - \tfrac{1}{2}t\tau^2 + \tfrac{1}{3}\tau^3) = t\tau^2 - \tfrac{1}{6}\tau^3 - 1 \geqslant 0$$

and

$$\int_{t-\tau}^t ds - \left(\tau - \frac{\tau^2}{t}\right) = \frac{\tau^2}{t} \geqslant 0.$$

Thus we conclude that any solution $x(t)$ of (a) with the property that $x(t)/t^2 \to 0$ as $t \to \infty$ is oscillatory. We note that Theorem 1 in [2] is not applicable since $a_1(t) \neq 1$.

THEOREM 2. *Suppose that*

$$\int_{t-\tau}^{t} \frac{1}{a_1(s_{n-1})} \int_{s_{n-1}}^{t} \cdots \int_{s_1}^{t} q(s)\, ds\, ds_1 \cdots ds_{n-1} > 1, \tag{12}$$

then every solution $x(t)$ *of* (2) *with the property that* $\lim_{t\to\infty}(x(t)/a_1(t)) = 0$ *is oscillatory.*

Proof. We only consider (2) when n is odd. Since a similar argument holds when n is even. Let $x(t)$ be a nonoscillatory solution of (2) such that $x(t)/a_1(t) \to 0$ as $t \to \infty$. Without any loss of generality we assume that $x(t)$ and $x(t-\tau)$ are positive for $t \geqslant t_0 > 0$. Then by Lemma 3 there is a $t_1 \geqslant t_0$ such that

$$\dot{x}(t) < 0 \quad \text{and} \quad (-1)^k L_k x(t) > 0 \quad \text{for} \quad t \geqslant t_1, \quad k = 2, \ldots, n-1.$$

If $s \leqslant t$, we have

$$x(s-\tau) \geqslant x(t-\tau).$$

Hence (2) implies that

$$L_n x(s) + q(s)x(t-\tau) \leqslant L_n x(s) + q(s)x(s-\tau) = 0. \tag{13}$$

Integrating the left-hand side of inequality (13) n times, we have

$$\begin{aligned} x(t) - x(t-\tau) &+ (-1)\, L_1 x(t) \int_{t-\tau}^{t} \frac{1}{a_1(s_{n-1})}\, ds_{n-1} + \cdots \\ &+ (-1)^{n-1} L_{n-1} x(t) \int_{t-\tau}^{t} \frac{1}{a_1(s_{n-1})} \int_{s_{n-1}}^{t} \cdots \\ &\times \int_{s_2}^{t} \frac{1}{a_{n-1}(s_1)}\, ds_1 \cdots ds_{n-1} + x(t-\tau) \\ &\times \int_{t-\tau}^{t} \frac{1}{a_1(s_{n-1})} \int_{s_{n-1}}^{t} \cdots \int_{s_1}^{t} q(s)\, ds\, ds_1 \cdots ds_{n-1} \leqslant 0, \end{aligned}$$

which implies that

$$\begin{aligned} x(t-\tau) &\geqslant x(t) + x(t-\tau) \int_{t-\tau}^{t} \frac{1}{a_1(s_{n-1})} \int_{s_{n-1}}^{t} \cdots \\ &\geqslant x(t-\tau) \int_{t-\tau}^{t} \frac{1}{a_1(s_{n-1})} \int_{s_{n-1}}^{t} \cdots \int_{s_1}^{t} q(s)\, ds \cdots ds_{n-1}. \\ &\times \int_{s_1}^{t} q(s)\, ds \cdots ds_{n-1} \end{aligned}$$

Thus

$$1 \geqslant \int_{t-\tau}^{t} \frac{1}{a_1(s_{n-1})} \int_{s_{n-1}}^{t} \cdots \int_{s_1}^{t} q(s)\, ds\, ds_1 \cdots ds_{n-1},$$

which contradicts (12). This contradiction completes the proof.

EXAMPLE 2. Consider the equation

(b) $$\dot{x}(t) + x(t - \tfrac{1}{2}\pi) = 0, \qquad t > 0.$$

$$\int_{t-\tau}^{t} ds = \tau = \tfrac{1}{2}\pi > 1.$$

Condition (12) is verified, and hence (b) is oscillatory.

EXAMPLE 3. Consider the equation

(c) $$(t^{-1}\dot{x})^{\cdot} - x(t-\tau) = 0, \qquad t > \tfrac{2}{3}\tau + (2/\tau^2).$$

Condition (12) is satisfied, since

$$\int_{t-\tau}^{t} s_1 \left(\int_{s_1}^{t} ds \right) ds_1 = \tfrac{1}{2}t\tau^2 - \tfrac{1}{3}\tau^3 > 1.$$

Thus by Theorem 2 every solution x of (c) with the property that $x(t)/t^2 \to 0$ as $t \to \infty$ is oscillatory.

EXAMPLE 4. Consider the equation

(d) $$(t^{-1}(t^{-1}\dot{x})^{\cdot})^{\cdot} + x(t-\tau) = 0.$$

Here

$$\int_{t-\tau}^{t} s_2 \int_{s_2}^{t} s_1 \int_{s_1}^{t} ds\, ds_1\, ds_2 = \left[\frac{1}{6} t^2 - \frac{5}{24} t\tau + \frac{1}{15} \tau^2 \right] \tau^3.$$

The above expression in t and τ is larger than 1 if $\tau \geqslant 2$.

The hypotheses of Theorem 2 are satisfied and thus we conclude that every solution $x(t)$ of (d) with the property that $x(t)/t^2 \to 0$ as $t \to \infty$ is oscillatory.

Remark. The above conclusions for Eqs. (a)–(d) are not deducible from Lada's result.

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